



## Extended Antipodal Theorems

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### Abstract

The paper presents the extensions of both the antipodal (Borsuk–Ulam) theorem and Browder theorem to the cases comprising both a star-shaped domain of the considered mapping and a multi-valued structure of that mapping. Moreover, an explicit algorithm constructing the desired connected path of the zero points of the mapping is developed.

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1            Since 1909 when Brouwer proved the first fixed-point theorem named after him, the fixed-  
point results in various settings play an important role in the optimization theory and applications.  
This technique has proven to be indispensable for the proofs of multiple results related to the  
existence of solutions to numerous problems in the areas of optimization and approximation theory,  
differential equations, variational inequalities, complementary problems, equilibrium theory, game  
theory, mathematical economics, etc. It is also worthwhile to mention that the majority of problems  
of finding solutions (zero-points) of functions (operators) can be easily reduced to that of  
discovering of fixed points of properly modified mappings.

2            Not only theoretical but also practical (algorithmic) developments are based on the fixed-  
point theory. For instance, the well-known simplicial (triangulation) algorithms help one to find the  
desired fixed points in a constructive way. That approach allows one to investigate the solvability of  
complicated problems arising in theory and applications.

3            In this extended abstract, by making use of the triangulation technique, we extend some  
antipodal and fixed-point theorems to the case of non-convex, more exactly, star-shaped sets. Also,  
similar extensions are made for set-valued mappings defined over star-shaped sets. From now on,  
we briefly explain the main points of our extensions.

4            The techniques using various fixed-point theorems have always been widely applied in  
Operations Research, Mathematical Programming, Game Theory, and other areas of optimization.  
Such techniques are appropriate in establishing the existence of solutions to mathematical  
programming problems, convex games, mathematical programs with equilibrium constraints  
(MPEC), to mention only few. Because of that, any extensions of the classical fixed-point theorems  
(like the Brouwer theorem for single-valued mappings and Kakutani theorem for multi-valued  
mappings) are interesting and important. In other words, topological tools are of a very high  
importance and use in the development of optimization theory and applications.

5            The Brouwer fixed-point theorem and the Borsuk-Ulam theorem can be characterized as  
two most powerful topological tools of extremely similar structure. Most topology textbooks that  
include these theorems, such as, for example, Krasnosel'skii [1], Herings [2], Yang [3], do not  
mention that the two are tightly related (for instance, the Borsuk-Ulam theorem implies the Brouwer  
Fixed Point Theorem).

6            Many fixed-point theorems have been established for the mappings defined on convex  
domains. However, in many applications and real-life problems, the domains need not be convex;  
for example, the feasible sets of bilevel programming problems mostly lack this property even for  
linear bilevel problems (*cf.*, [4]). In this abstract, we outline how we extend the Borsuk-Ulam  
(antipodal) theorem to more general domains and to the case of multi-valued mappings as well.

7            First, we recall the definition of star-shaped sets and some properties of projections onto  
them.

8            **Definition 1.** *A star-shaped region centered at  $x_0$  in  $R^n$  is a set  $D \subset R^n$  such that for every*  
 *$x \in D$  and  $t \in [0,1]$  one has  $\gamma_x(t) := x_0 + t(x - x_0) \in D$ .*

9            Consider  $R^n$  with the maximum metric:

10             $d_\infty(x, y) := \max \{|x_i - y_i| : 1 \leq i \leq n\}$ , where  $x, y \in R^n$ ,

11            as well as the norm induced by this metric:

12

$$\|x\|_{\infty} := \max \{ |x_i| : 1 \leq i \leq n \}, x \in \mathbb{R}^n.$$

13 In order to work with the projections of the points inside and outside a star-shaped subset onto its boundary, we need the following notation.

14 **Definition 2.** Let  $D$  be a closed, bounded, and star-shaped region  $D$  centered at  $x_0=0$ . For any  $r>0$  such that  $D \subset C^n(r) := \{x \in \mathbb{R}^n : \|x\|_{\infty} \leq r\}$  we define the following:

15 • For  $x \in C^n(r) \setminus D$  define  $\theta_x := \max \{t \in [0,1] : \gamma_x(t) \in D\}$ , and  $\alpha(x) := \gamma_x(\theta_x)$ .

16 • For  $y \in D$ , let  $u_y := \inf \{t \in [1,+\infty) : \gamma_y(t) \in D\}$ , and  $\lambda(y) := \gamma_y(u_y)$ .

17 • For  $z \in C^n(r)$ , let  $s_z := \inf \{t \in [1,+\infty) : z \in C^n(r)\}$ , and  $\omega(z) := \gamma_z(s_z)$ .

18 The antipodal property is the key feature of the functions examined in the Brouwer and Borsuk-Ulam theorems.

19 **Definition 3.** Let  $D \subset C^n(r)$  be a closed star-shaped set and  $\rho: D \rightarrow \mathbb{R}^n$  a continuous function. The function is said to have the antipodal property, if for every  $x \in \partial D$  it holds that  $\rho(x) = -\rho(\lambda(-x))$  if  $(-x) \in D$ , and  $\rho(x) = -\rho(\alpha(-x))$  whenever  $(-x) \notin D$ .

20 The well-known extension of the antipodal property is the nonparallel condition defined below.

21 **Definition 4.** Let  $D \subset C^n(r)$  be a closed star-shaped set and  $\rho: D \rightarrow \mathbb{R}^n$  a continuous function. This function satisfies the nonparallel condition if for every  $x \in \partial D$  the following relationships are true: for all  $c \geq 0$ , one has  $\rho(x) \neq c\rho(\lambda(-x))$  if  $(-x) \in D$ , and  $\rho(x) \neq c\rho(\alpha(-x))$  otherwise, that is, whenever  $(-x) \notin D$ .

22 We say that  $\rho: D \rightarrow \mathbb{R}^n$  satisfies a weak nonparallel condition if for all  $c > 0$ , it holds:  $\rho(x) \neq c\rho(\lambda(-x))$  if  $(-x) \in D$ , and  $\rho(x) \neq c\rho(\alpha(-x))$  otherwise, that is, whenever  $(-x) \notin D$ .

23 Similar to the constructive proofs presented in [5 – 6], we deduce important existence results concerning zero points.

24 **Theorem 1.** Let  $D \subset C^n(r)$  be a closed star-shaped set and  $\rho: D \rightarrow \mathbb{R}^n$  a continuous function that satisfies the weak nonparallel condition. Then there exists a point  $x^* \in D$  such that  $\rho(x^*) = 0$ .

25 The above results is now extended to multi-valued mappings. In order to do that, we refer to the distance between sets as introduced in [8] – [10].

26 **Definition 5.** Let  $A, B$  be arbitrary subsets of  $\mathbb{R}^n$ . The distance between these subsets is defined as follows:

27

$$d(A, B) := \inf \{ \|x - y\| : x \in A, y \in B \}.$$

28 Now we can extend the antipodal and nonparallel conditions to the case of multi-valued mappings defined on star-shaped sets.

29 **Definition 6.** Let  $D \subset C^n(r)$  be a closed star-shaped set and  $\rho: D \rightarrow 2^{\mathbb{R}^n}$  a multi-valued mapping. The mapping is said to have the antipodal property if for every  $x \in \partial D$ , one has  $\rho(x) = -\rho(\lambda(-x))$  if  $\lambda(-x) \in D$ , and  $\rho(x) = -\rho(\alpha(-x))$  whenever  $\lambda(-x) \notin D$ ; here,  $-\rho(w) := \{-z : z \in \rho(w)\}$ .

30 **Definition 7.** Let  $D \subset C^n(r)$  be a closed star-shaped set and  $\rho: D \rightarrow 2^{\mathbb{R}^n}$  a multi-valued mapping. The mapping satisfies the nonparallel condition if for every  $x \in \partial D$  and  $c \geq 0$ , it holds  $\|y_1 - cy_2\| > 0$  for all  $y_1 \in \rho(x)$  and  $y_2 \in \rho(\lambda(-x))$  whenever  $\lambda(-x) \in D$ . Otherwise, if  $\lambda(-x) \notin D$  then  $\|y_1 - cy_2\| > 0$  for any  $y_1 \in \rho(x)$  and  $y_2 \in \rho(\alpha(-x))$ . If the above inequalities are claimed for (only)  $c > 0$  we say that  $p$  satisfies the weak nonparallel condition.

31 Now first, we demonstrate that the problem of finding zeros of a multi-valued mapping defined on a closed bounded star-shaped subset  $D$  of  $\mathbb{R}^n$  satisfying the nonparallel condition can be reduced to the problem of finding zeros of an extended multi-valued mapping boasting the antipodal property on some cube  $C^n(r)$ .

32 **Lemma 1.** Let  $D \subset C^n(r)$  be a closed star-shaped set and  $\rho: D \rightarrow 2^{\mathbb{R}^n}$  a multi-valued mapping satisfying the weak nonparallel condition. Then can be extended to a mapping  $\Psi: C^n(r) \rightarrow 2^{\mathbb{R}^n}$  with the antipodal property and such that  $x^* \in D$  whenever  $(0, \dots, 0) \in \Psi(x^*)$ .

33 Now the main result obtained for the multi-valued mappings on star-shaped set follows.

34 **Theorem 2.** Let  $\rho: D \subset C^n(r) \rightarrow 2^{\mathbb{R}^n}$  be an upper-semicontinuous mapping defined on a star-shaped set  $D$  and satisfying the weak nonparallel condition. Then there exists a point  $x^* \in D$  such that  $0 \in \rho(x^*)$ .

35 The abstract finishes with an extension of the nonparallel condition-related result for mappings defined on a star-shaped set times an interval  $D \times [0, 1]$  and having convex compact subsets of  $\mathbb{R}^n$  as its values. Moreover, we will study a connected set of zero points intersecting both  $D \times \{0\}$  and  $D \times \{1\}$ , in order to extend the Browder theorem.

36 **Theorem 3.** Let  $\rho: D \times [0, 1] \rightarrow \mathfrak{I}(\mathbb{R}^n)$  be a multi-valued mapping from  $D \times [0, 1]$  to  $\mathfrak{I}(\mathbb{R}^n)$  where  $\mathfrak{I}(\mathbb{R}^n) := \{A \subset \mathbb{R}^n : A \text{ is a convex set}\}$  and  $D$  be a compact star-shaped subset of  $\mathbb{R}^n$  with respect to the origin. Suppose that the following conditions hold for  $p$ :

37 1. The mapping is upper-semicontinuous.

38 1. For every  $t \in [0, 1]$ , the restriction  $\rho_t: D \rightarrow 2^{\mathbb{R}^n}$  defined by  $\rho_t(x) := \rho(x, t)$  satisfies the nonparallel condition. Then there exists a connected subset  $Z$  in  $D \times [0, 1]$  such that  $Z \cap (D \times \{0\}) \neq \emptyset$ ,  $Z \cap (D \times \{1\}) \neq \emptyset$ , and whenever  $(x, t) \in Z$  then  $0 \in p(x, t)$ .

39 The paper presents the extensions of both the antipodal (Borsuk–Ulam) theorem and Browder theorem to the cases comprising both a star-shaped domain of the considered mapping and a multi-valued structure of that mapping. Moreover, an explicit algorithm constructing the desired connected path of the zero points of the mapping is developed.

40 The properties of the latter algorithm are used in the proof of the extended Browder Theorem, similar to the existence proofs in the pioneering papers by Todd [7], van der Laan and Talman [5] – [6].

41 In our future research, we are going to extend the above-mentioned results to other classes of nonconvex domains of the examined multi-valued mappings.

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# Обобщения антиподальных теорем

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## Аннотация

В статье доказаны обобщения теоремы об антиподальности (Борсука-Улама) и теоремы Браудера на случай звездной структуры множества определения рассматриваемого многозначного отображения. Кроме того, разработан прямой алгоритм построения в явном виде связной кривой, содержащей нули этого отображения.

**Ключевые слова:** Множества звездной структуры, условие антиподальности, условие непараллельности.

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